Brownian Motion - The Brownian Bridge
Part I - Base Equations

Gary Schurman, MBE, CFA

August, 2021

In this white paper we will build a Brownian bridge (i.e. a random path) between two known end points. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are tasked with building a Brownian bridge between a start and end point given the following model parameters...

Table 1: Model Parameters

Description	Value
Value of the Brownian motion at time zero	0.00
Value of the Brownian motion at time T	1.35
Time in years (T)	2.00

Task: Graph the mean and variance of the random path between the Brownian motion's start and end points.

Building Our Model

We will define the variable t to be time in years where t can be any value in the time inteval $[t(a), t(b)]$. We will define the variable X_t to be the value of the variable X at time t. The variable X is defined as the sum of a deterministic (i.e. non random) function and a random function. The deterministic function is a function of time and the random function is the change in an underlying Brownian motion.

We are given the values of X at the beginning and end of the time interval $[t(a), t(b)]$ and are asked to simulate the random path between the start point and end points. We will define the start point $X_{t(a)}$ and end point $X_{t(b)}$ to be...

$$a = X_{t(a)} \text{ ...and... } b = X_{t(b)} \tag{1}$$

We will define the variable W_t to be the value of the Brownian motion (the random part of X_t) at time t. Using Equation (1) above the equation for the stochastic differential equation (SDE) that defines the change in X_t over time is...

$$\delta X_t = \frac{b - X_t}{t(b) - t}\,\delta t + \delta W_t \text{ ...where... } \delta W_t \sim N\Big[0, \delta t\Big] \text{ ...and... } t \in \Big[t(a), t(b)\Big] \tag{2}$$

Note that we can rewrite Equation (2) above as...

$$\delta X_t + \frac{X_t}{t(b) - t}\,\delta t = \frac{b}{t(b) - t}\,\delta t + \delta W_t \tag{3}$$

If we multiply Equation (3) above by $\frac{1}{T-t}$ then that equation becomes...

$$\frac{\delta X_t}{t(b) - t} + \frac{X_t}{(t(b) - t)^2}\,\delta t = \frac{b}{(t(b) - t)^2}\,\delta t + \frac{1}{t(b) - t}\,\delta W_t \tag{4}$$

Using Appendix Equation (20) below we can rewrite Eqation (4) above as...

$$\delta\left(\frac{1}{t(b) - t}\,X_t\right) = \frac{b}{(t(b) - t)^2}\,\delta t + \frac{1}{t(b) - t}\,\delta W_t \tag{5}$$

Integrating both sides of Equation (5) above we get the following equation...

$$\int\limits_{t(a)}^{t} \delta\left(\frac{1}{t(b)-u}\,X_u\right) = \int\limits_{t(a)}^{t} \frac{b}{(t(b)-u)^2}\,\delta u + \int\limits_{t(a)}^{t} \frac{1}{t(b)-u}\,\delta W_u \tag{6}$$

The solution to Equation (6) above is...

$$\frac{1}{t(b)-u}\,X_u \Bigg[_{u=t(a)}^{u=t} = \frac{b}{t(b)-u} \Bigg[_{u=t(a)}^{u=t} + \int\limits_{t(a)}^{t} \frac{1}{t(b)-u}\,\delta W_u \tag{7}$$

After applying the bounds of integration to the Equation (7) above the Equation for X_t becomes...

$$\frac{1}{t(b)-t}\,X_t - \frac{1}{t(b)-t(a)}\,X_{t(a)} = \frac{b}{t(b)-t} - \frac{b}{t(b)-t(a)} + \int\limits_{t(a)}^{t} \frac{1}{t(b)-u}\,\delta W_u$$

$$\frac{1}{t(b)-t}\,X_t = \frac{a}{t(b)-t(a)} + \frac{b}{t(b)-t} - \frac{b}{t(b)-t(a)} + \int\limits_{t(a)}^{t} \frac{1}{t(b)-u}\,\delta W_u$$

$$\frac{1}{t(b)-t}\,X_t = \frac{b}{t(b)-t} + \frac{a-b}{t(b)-t(a)} + \int\limits_{t(a)}^{t} \frac{1}{t(b)-u}\,\delta W_u$$

$$X_t = b + \frac{(a-b)\,(t(b)-t)}{t(b)-t(a)} + (t(b)-t)\int\limits_{t(a)}^{t} \frac{1}{t(b)-u}\,\delta W_u \tag{8}$$

Mean and Variance

The expectations applicable to the change in the underlying Brownian motion are...

$$\mathbb{E}\Bigg[\delta W_t\Bigg] = 0 \text{ ...and... } \mathbb{E}\Bigg[\delta W_t^2\Bigg] = \delta t \text{ ...and... } \mathbb{E}\Bigg[\delta W_t\,\delta W_u\Bigg] = 0 \tag{9}$$

Using Equations (8) and (9) above the equation for the mean of the random variable X_t is...

$$\text{mean} = \mathbb{E}\Bigg[b + \frac{(a-b)\,(t(b)-t)}{t(b)-t(a)} + \frac{1}{t(b)-t}\int\limits_{t(a)}^{t} \frac{1}{t(b)-u}\,\delta W_u\Bigg] = b + \frac{(a-b)\,(t(b)-t)}{t(b)-t(a)} \text{ ...because... } \mathbb{E}\Bigg[\delta W_u\Bigg] = 0 \tag{10}$$

Using Equations (8) and (9) above the equation for the variance of the random variable X_t is...

$$\text{variance} = \mathbb{E}\Bigg[\Bigg((t(b)-t)\int\limits_{t(a)}^{t} \frac{1}{t(b)-u}\,\delta W_u\Bigg)^2\Bigg] = \mathbb{E}\Bigg[(t(b)-t)^2 \int\limits_{u=t(a)}^{u=t}\int\limits_{v=t(a)}^{v=t} \frac{1}{t(b)-u}\,\frac{1}{t(b)-v}\,\delta W_u\,\delta W_v\Bigg] \tag{11}$$

Note that since $\mathbb{E}[\delta W_t\,\delta W_u] = 0$ we can ignore those cases are rewrite Equation (11) above as...

$$\text{variance} = \mathbb{E}\Bigg[(t(b)-t)^2 \int\limits_{t(a)}^{t} \frac{1}{(t(b)-u)^2}\,\delta W_u^2\Bigg] = (t(b)-t)^2 \int\limits_{t(a)}^{t} \mathbb{E}\Bigg[\frac{1}{(t(b)-u)^2}\,\delta W_u^2\Bigg] = (t(b)-t)^2 \int\limits_{t(a)}^{t} \frac{1}{(t(b)-u)^2}\,\delta u \tag{12}$$

Using Appendix Equation (23) below the solution to Equation (12) above is...

$$\text{variance} = (t(b)-t)^2\,\frac{1}{t(b)-u}\Bigg[_{u=t(a)}^{u=t} = (t(b)-t)^2\left(\frac{1}{t(b)-t} - \frac{1}{t(b)-t(a)}\right) = \frac{(t(b)-t)\,(t-t(a))}{t(b)-t(a)} \tag{13}$$

The Answer To Our Hypothetical Problem

Using Equation (1) above and the parameters in Table 1 above we can make the following definitions...

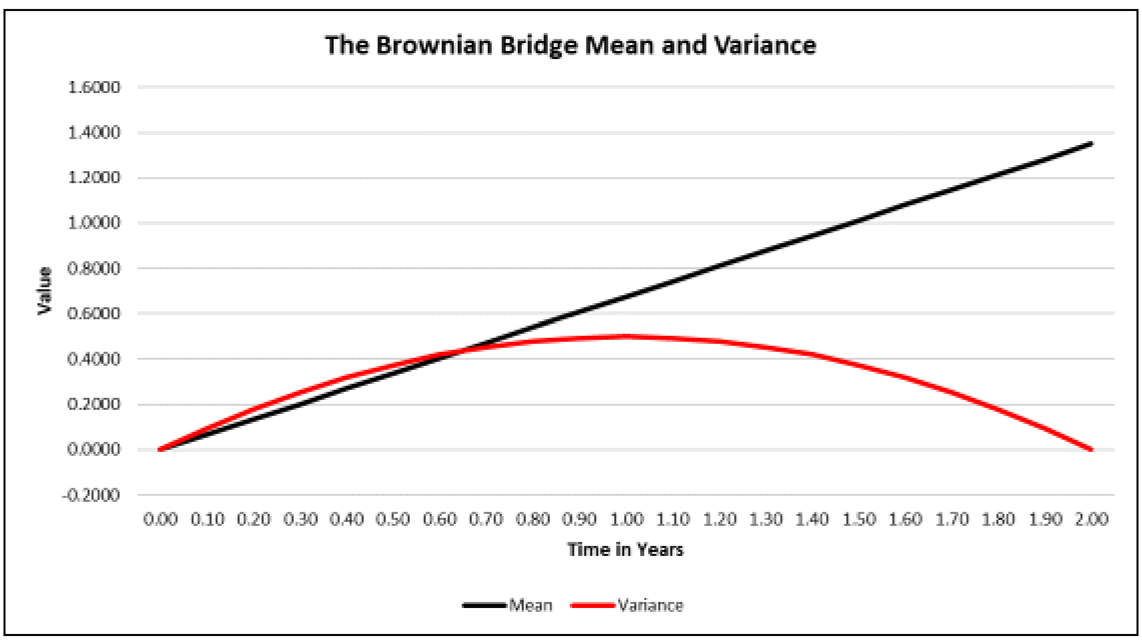
$$a = 0.00 \text{ ...and... } b = 1.35 \text{ ...and... } t(a) = 0 \text{ ...and... } t(b) = 2 \tag{14}$$

Using the definitions in Equation (17) above our Brownian bridge mean (Equation (10)) and variance (Equation (13)) between $X_{t(a)}$ and $X_{t(b)}$ is...

Table 2: Random Path Mean and Variance

A	B	C	D
t	δt	mean	variance
0.00	–	0.0000	0.0000
0.10	0.10	0.0675	0.0950
0.20	0.10	0.1350	0.1800
0.30	0.10	0.2025	0.2550
0.40	0.10	0.2700	0.3200
0.50	0.10	0.3375	0.3750
0.60	0.10	0.4050	0.4200
0.70	0.10	0.4725	0.4550
0.80	0.10	0.5400	0.4800
0.90	0.10	0.6075	0.4950
1.00	0.10	0.6750	0.5000
1.10	0.10	0.7425	0.4950
1.20	0.10	0.8100	0.4800
1.30	0.10	0.8775	0.4550
1.40	0.10	0.9450	0.4200
1.50	0.10	1.0125	0.3750
1.60	0.10	1.0800	0.3200
1.70	0.10	1.1475	0.2550
1.80	0.10	1.2150	0.1800
1.90	0.10	1.2825	0.0950
2.00	0.10	1.3500	0.0000

Our graph of the Brownian bridge mean and variance is...



Appendix

A: The derivative of the following equation with respect to time via the quotient rule is...

$$\begin{aligned}\frac{\delta}{\delta t}\left(\frac{1}{T-t}\right) &= \left[\frac{\delta}{\delta t}(1) \times (T-t) - \frac{\delta}{\delta t}(T-t) \times 1\right] \Big/ (T-t)^2 \\ &= 0 \times (T-t) - \frac{1}{(T-t)^2} \times -1 \\ &= \frac{1}{(T-t)^2} \end{aligned} \tag{15}$$

B: The derivative of the following equation via the product rule is...

$$\delta\left(\frac{1}{T-t}\,X_t\right) = \delta\left[\frac{1}{T-t}\right] \times X_t + \delta\left[X_t\right] \times \frac{1}{T-t} \tag{16}$$

Using Appendix Equation (18) above we can rewrite Equation (19) above as...

$$\delta\left(\frac{1}{T-t}\,X_t\right) = \frac{\delta X_t}{T-t} + \frac{X_t}{(T-t)^2}\,\delta t \tag{17}$$

C: We want to solve the following integral...

$$\int_0^t \frac{1}{(T-u)^2}\,\delta u \tag{18}$$

Using Equation (18) above the anti-derivative of the integrand in Equation (21) above is

$$\text{Anti-derivative of } \frac{1}{(T-t)^2} = \frac{1}{T-t} \tag{19}$$

Using Equation (22) above the solution to the integral in Equation (21) is...

$$\int_0^t \frac{1}{(T-u)^2}\,\delta u = \frac{1}{T-u}\Bigg[_{u=0}^{u=t} \tag{20}$$